Semantic Warrants, Mathematical Referents, and Personal Agency in Theory Building

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Abstract. We examine university honors calculus students' collaborative development of mathematical methods for finding the volume of a solid of revolution. We qualitatively analyze students' semantic warrant productions in substantial argumentation during public performances. Students chose specific mathematical referents in the production of solution approaches generated during extended problem solving. Students were convinced of the reasonableness of multiple solution approaches through semantic warrant production during public performances over time and were strongly influenced by the introduction of the First Theorem of Pappus after they invented the theorem in response to mathematical necessity in problem solving. Students' enactments of personal agency were generative for semantic warrant production and grounded the logical structure of students' substantial arguments. This study contributes to the literature on the strengths of students' authentic mathematics creativity within a task-based classroom setting wherein enactments of personal agency are mathematically generative.

Introduction. Student difficulties in justification, argumentation, and proof have been well documented in the literature (Ellis, 2007). Few studies, however, have carefully analyzed the mathematically creative processes involving justification and argumentation amongst groups of students building theory in calculus. Here, we examine the inventive development of the First Theorem of Pappus by honors calculus students as they work without prior instruction in collaborative groups and in public performances to find the volume of a solid of revolution. Our analysis highlights honors calculus students' semantic warrant productions in substantial argumentation during public performances and contributes to the literature on the strengths of students' authentic mathematics creativity within a task-based classroom setting wherein enactments of personal agency are mathematically generative. Central to our analysis are the roles of personal agency and performance in students' interactions to creatively find the volume of a solid of revolution, and semantic warrant production in substantial argumentation toward building consensus.

Related Literature. In his classic text challenging mathematical formalism, Lakatos (1976) writes, "informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (p. 5). Our study characterizes the emergence of a special case of the First Theorem of Pappus through students' incessant efforts at making sense of a novel context to find the volume of a solid of revolution.

In a study of undergraduate, doctoral students', and mathematicians' formal proof productions, Weber and Alcock (2004) noted that undergraduates worked strictly syntactically in terms of mathematical definitions which required modest knowledge and abilities. Mathematicians, on the other hand, worked semantically with global or intuitive observations translated into formal reasoning which required rich, accurate instantiations connected with concepts and suggested inferences. As a result, Weber and Alcock distinguished between syntactic and semantic proof production, "…one who produces a proof syntactically may be said to understand what to do, and one who produces a proof semantically may be said to understand both what to do and why" (p. 231).

Our goal is to build contextual theory regarding the structure and function of undergraduates' substantial argumentation in calculus when argumentation is directed toward mathematical consensus in the learning community rather than explicitly toward formal proof production.

Four mechanisms for supporting students' engagement in reasoning and the mutually influential relationship between generalizing and justifying were explored in a study of middle school students' development of justification (Ellis, 2007). Findings from the study suggested that "actions that may be considered unacceptable from an expert's perspective could be the very ones that ultimately support more appropriate outcomes" (Ellis, 2007, p. 225). Although the work of mathematicians, as evidenced in the Weber and Alcock (2004) study, is initially semantic and intuitive, students in mathematics courses are often discouraged from working similarly, and are instead encouraged by some experts to produce syntactic justifications or to employ syntactic applications of extant algorithms.

Cooley, Trigueros, and Baker (2007) interviewed college students who had successfully completed multivariate calculus to demonstrate thematization of students' calculus graphing schema and provided a retrospective account of students' individual reconstruction of mathematical ideas previously learned under indeterminate conditions. The construction or reconstruction of ideas by students transitioning from concrete or empirical meanings toward abstraction (Pijls, Dekker & Van Hout-Wolters, 2007; Rivera, 2007), depend, in part, on giving explanations and criticizing oneself. However, for meaningful sense making and to gain understandings of others' ways of knowing within a learning community, students must engage in behaviors consistent with the exercise of personal agency in playing the believing game (Elbow, 1973).

Recently, Walter & Johnson (2007) have suggested that development of mathematical meaning may emerge in learners' production of semantic warrants. Semantic warrants are personally meaningful instantiations that ground developmental reasoning and support mathematical inferences. The producers of semantic warrants are intrinsically motivated to make purposeful choices, within the social milieu of the learning community, to offer semantic warrants for consideration or validation by others. Semantic warrant production occurs during substantial argumentation (Toulmin, 1969) directed toward building consensus regarding mathematical understandings among members of a learning community. Substantial argumentation is "the accomplishment of a convincing presentation of backgrounds, relations, explanations, justification, qualifiers, and so on" (Krummheuer, 1995, p. 236). Learners "purposefully choose to engage in semantic warrant production in order to convince themselves and others of the correctness of personally constructed mathematics" (Walter & Johnson, 2007, p. 709). The mathematical work of students engaged in substantial argumentation may differ greatly. Indeed, Goffman (1959) recognized that "working consensus established in one interaction setting will be quite different in content from the working consensus established in a different type of setting" (p. 10).

Theoretical Perspective. Our theoretical perspective is threefold and grounded in personal agency, performance, and interaction. The exercise of personal agency within a community of learners is made evident as students act and interact with volition. We agree with a perspective of agency which holds that in order to act a person must be able to make choices based on perception and available information (Bandura, 1997; Holland, Skinner, Lachicotte Jr, Cain, & Delmouzou, 1998; Skovsmose, 2005). Acts of purposeful choice are fundamental to learning (Brown, 2005; Kohn, 1998; Rogers, 1969; Walter & Gerson, 2007). The choices students make are creative acts essential for the development of mathematical meaning, thinking and learning.

Powell (2004) suggests that agency in mathematical learning may be characterized as learners' individual initiative to define, redefine, build on or go beyond specificities of mathematical situations on which they have been invited to work. Bandura (1997) views emergent interactive agency as causative interaction based on self-efficacy beliefs toward affecting change. Bratman (2007) suggests that agency may include individual self-governance, intention, planning, shared agency, and temporally extended agential authority.

Indeed, the exercise of personal agency is the genesis of creative acts that shape, and in turn are influenced within, the milieu of lived experience. In particular, for us, personal agency is "the requirement, responsibility and freedom to choose based on prior experiences and imagination, with concern not only for one's own understandings of mathematics, but with mindful awareness of the impact one's actions and choices may have on others… Because people build understanding from experience, it is essential that they have opportunities to make personal choices that will foster learning in particular, perhaps unanticipated, ways as they explore mathematics and develop a sense of self as actor and participant. The exercise of agency is what makes mathematical thinking possible" (Walter & Gerson, 2007, p. 209).

Goffman (1959) described performance as "all the activity of a given participant on a given occasion which serves to influence in any way any of the other participants" (Goffman, 1959, p. 15). We suggest that performance in a learning community may be characterized as "an observable, flexible, synchronous process of reasoning, presenting and organizing one's thoughts…[and] begins with an individual choosing to act, which may influence and include actions of the group" (Walter & Gerson, 2007, p. 206). Interaction has been defined as "the reciprocal influence of individuals upon one another's actions when in one another's immediate physical presence" (Goffman, 1959, p. 15) and "a flowing process in which each participant is guiding his action in the light of the action of the other suggests its many potentialities for divergent direction" (Blumer, 1969, p. 110).

In student-centered, investigatory task-based classrooms, all three grounding attributes of our theoretical perspective are elemental in student endeavors. Against such a background, we arrive at our initial research question. In the process of a grounded theory approach to data collection and analysis, several additional questions emerged.

Initial Research Question: How do university honors calculus students collaboratively develop mathematical methods for finding the volume of a solid of revolution when no prior instruction on solution methods was given?

Emergent Questions: How do students use semantic warrants in substantial argumentation during public performances? What logical argumentation structures are evident in calculus students' production of substantial arguments?

Methodology. Eighteen second-semester honors calculus students, comprising four groups, worked collaboratively during two-hour class sessions three times a week on open-response tasks. Tasks were designed to foster creativity while eliciting conceptually important mathematics as part of a 3-semester teaching experiment at a large private university in the mountain-west region of the United States. Each semester-long course was team-taught by the principle investigators and pedagogical decisions were based on the progress and direction of inquiry by students, rather than by textbook organization. The data presented here were collected during three 2-hour class sessions during the second month of the semester. During these sessions, students worked collaboratively on one task, the Gel-Pack Mug Task (Figure 1) without prior instruction on solution approaches. The task was designed by the principal investigators to elicit student generated, meaningful strategies for finding the volume of a solid of revolution.

Second-semester calculus students have prior experience with integration techniques and applications, but finding volumes of solids of revolution is a new topic. Our analysis here will focus on students' public performances of their small-group invented solutions for the following task.

> A cold mug consists of a gel pack sandwiched by two cylinders. For manufacturing reasons, we make the gel pack parabolic in cross-section. We are interested in knowing the volume of the gel. For your information, the height of the gel pack is 12 cm. The gel is filled up to the 11 cm mark. The top of the gel pack needs to be 2 cm wide at the top. The inside radius of the mug is 8 cm.

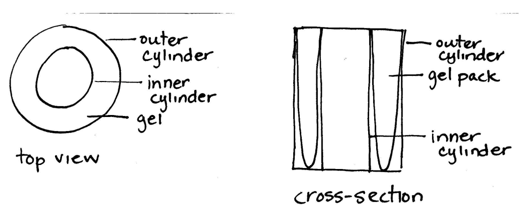

Figure 1. Gel-Pack Mug Task

Each class session was videotaped. Transcripts of videotaped student discourse were verified by research team members which included graduate and undergraduate mentored students. Transcripts were memoed to record insights and relationships between events and annotated to reflect gestures, intonations, pauses, and other student activity during student conversations. Constant comparison of all data sources, including student written work and notations developed over time, homework assignments and finished task write-ups, substantiated video analysis.

Coding. Open and axial coding (Strauss & Corbin, 1998) during microlinguistic analysis characterized mathematical referents in semantic warrants and structures of substantial arguments. Open coding included repeated searches of the transcript for student language that indicated student attempts at justification, argumentation, consensus, and proof. An coding map produced by the qualitative research software Transana, is presented below (Figure 2). The map indicates with arbitrary colors when a speaker stated a particular word during the videotaped session and is useful in pattern and frequency analyses. Axial coding of students substantial arguments revealed pivotal cues in students' arguments. Pivotal cues included words such as for, because, but, like, mean, proof, prove, so, and then, and were incorporated into the coding map. Patterns in students' use of pivotal cues were examined. Mathematical referents surfaced in students' substantial arguments during open and axial coding. Mathematical referents are taken-as-shared mathematical objects to which students appeal and may be personal instantiations in semantic warrants or used as backing in substantial argumentation.

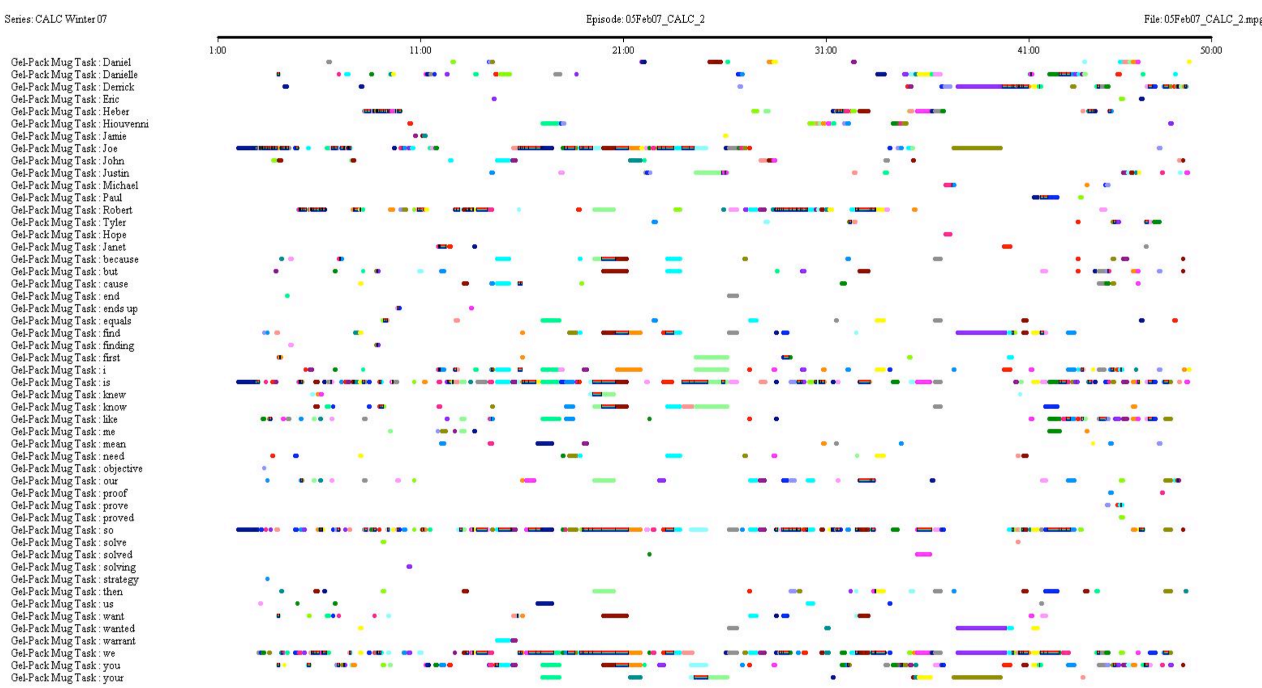

Figure 2. Coding Map

Transcript excerpts presented here are from the three days that students worked on the task. Gestures, discourse clarification, and analysis are in brackets within the transcript. Time codes in minutes:seconds provide for reference to particular events and a sense of temporal breadth.

Data and Analysis. First, we briefly characterize four creative solution approaches which were collaboratively built by students while working in their groups and which were shaped in public performances (Author, 2007). Then, we focus on the two solution approaches which, together, appropriately comprise a special case of the First Theorem of Pappus.

Four Solution Approaches. All four groups came to equivalent conclusions, via different approaches, that the volume of the gel-pack was 264π cubic centimeters. Two groups chose to use what might be considered traditional calculus methods to find the desired volume. Students seated at Table 1 (T1) chose to utilize the shell method and students at Table 3 (T3) chose to approach the problem with a strategy conventionally called the disc method. Students seated at Table 2 (T2) or at Table 4 (T4) invented distinct, unconventional methods for solving the task. T2 students developed what they termed to be the "ratios" method. In the ratios method, after integrating to determine the area of one of the parabolic faces represented in the cross-section of the gel-pack mug, the group calculated the ratio of the parabolic area to the area of the rectangle circumscribing the parabola. Then they multiplied that ratio by the volume of the cylinder or "shell containing the gel-pack" to find the volume of the gel-pack (Figure 3). Students at Table 4 (T4) invented a method which is a special case of the First Theorem of Pappus. In this paper, we focus on the mathematical discourse during public performances by students from T3 and T4. We chose this focus because of the personal agency enactments and interactions between students in these two groups during public performances that collaboratively supported students' invention of the First Theorem of Pappus.

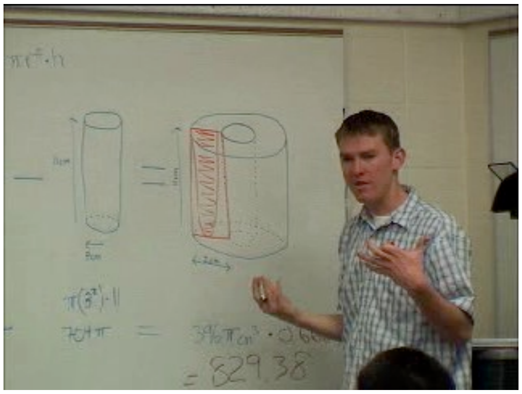

Figure 3. Justin's presentation of the ratios method

Table 3. Four students were seated at T3. Robert, Michael, and Heber were enrolled for the first time in the experimental calculus project. Joe completed calculus one in the experimental project the previous semester. Joe and Robert explained to the class the initial development of the disc method solution by T3. Members of the class asked questions regarding the use and meaning of different variable notations. In the following excerpt (D2 12:46), Robert characterized the chronology of purposeful choices in choosing distinct variables in the group's notations.

D2 12:46 Robert: I guess kinda to go in like time chronological of how we thought about it, at least for me individually [personal agency] Uh, I looked at the curve represented by the gel [points to the graph of the parabola with vertex located at the origin] and came up with an equation for that [process] which is y equals eleven x squared [what we have/mathematical referent] and so that has it's own origin [points to the vertex of the curve $y = 11x^2$ and hand sweeps along the curve] separate from the [graph of the parabolas for the gel in the] actual cup itself [points to the curve of the gel-pack mug]. So this [parabola at origin] this represents, at least for me, this represents just what one part of the gel looks like [points to the equation $y = 11x^2$], what that curve is represented by mathematically. And then looking at the big picture [indicates gel-pack cross section students sketched on axes system, Figure 4 below], we started dealing with r's, cause we're dealing with a physical object [cause/semantic warrant]. Uh we have two r's [what we have]. One is from the very center of the cup [points to origin] to the very inside of the gel [points to left branch of parabola with vertex at (8,0)] and the other is from the very inside of the cup to the very outside of the gel [points to right branch of parabola with vertex at (8,0)] [backing] and so that's why we end up with r's and x's [so/conclusion]. So we keep this separate [points to parabola and $y = 11x^2$] [so/conclusion]. I don't know if that clarifies it or helps.

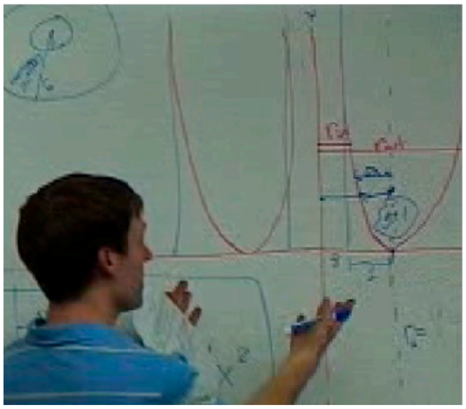

Figure 4. The Big Picture

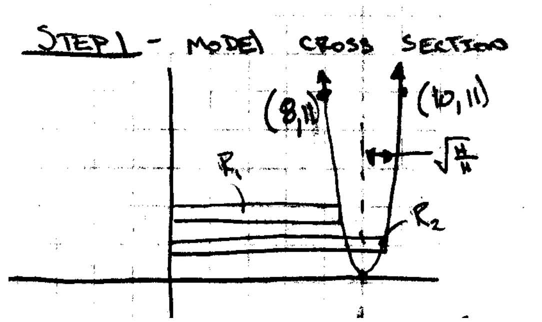

Figure 5. Michael's Homework Graph

Students at T3 chose to use the graph and associated equation of a parabola with vertex at the origin, the prototypical parabola $y = ax^2$, as mathematical referents in semantic warrants for further inferences regarding gel-pack graphs, equations, and volume. T3 placed the cross section of the gel-pack on axes to explore its characteristics (Figure 5). Robert justified using *r*, instead of *x*, to describe the radius of the gel-pack because they were "dealing with a physical object." The logical structure of Robert's substantial argument may be represented with the following: like-we-me (I)-points (gestures to referent)-so-points-so-me-like-then-we-'cause-we-so-we-so-we and interpreted as What We Have, What We Want, So/process, Semantic Warrant, So/process, So/conclusion. In a subsequent performance (D2 19:54), the logical structure of Robert's substantial argument may be characterized as: points (gesture)-we knew (given)-because (due to the fact that)-so (result)-so (in the way that follows)-points-so (stated as true)-because (due to the fact that)-then (as a consequence, inference or conclusion).

T3 continues their presentation by finding a representation for the volume of a representative disc. They integrate over an interval representing the height of the gel-pack to find the volume of the gel-pack.

Table 4. Four students are seated at Table 4: Danielle, Derrick, Paul and Nels. Derrick, Danielle, and Paul present their group's solution to the task. The mathematical referent in T4's

substantial argument is the formula for the volume of a prism, base area multiplied by the height of the prism (Figure 6). In the gel-pack mug task, T4 determines the area of the parabolic base and proceeds to find the third dimension in a compelling manner. The structure of the substantial argument crafted by Derrick and Paul is presented in the following transcript [D1: 00:41:15].

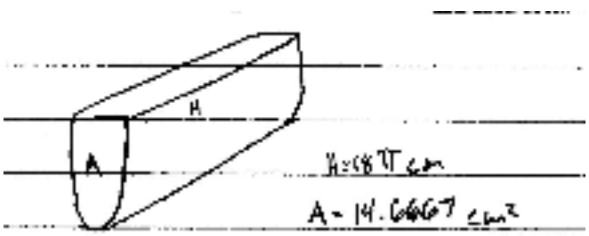

Figure 6. Derrick’s Parabolic Prism

D1: 00:41:15 Derrick: Is everybody clear on that? Yep. Okay, so we decided that since we have an area [of the parabolic face, points to area calculation: 14.6667], to find a volume [what we want] usually you find an area and multiply it times something else to get that third dimension that we wanted to do [process/mathematical referent]. So... Do you want to explain it? [shared agential authority]
00:41:30 Paul: Sure. So the third dimension, that we were going to find um if this [points to and motions across the parabolic region or base] was in rectangles it'd just be width times height [horizontal gesture from -1 to 1, vertical gesture from 0 to 11], would give us this area [so/semantic warrant]. And um, but our area is just the area of a parabola [what we have]. We want to find another dimension on top to multiply it [parabolic face area] by [sweeps hand forward pointing away from his body] to find the whole volume around the gel pack [points upward with circular gestures] [process]. So in order to do that we know the radius from the inside to the drinking cup part is eight and then the inside to the very outside is ten [points to and motions across the top view of the gel-pack] [semantic warrant] and so right in between that is nine [so/conclusion]. Um, we multiplied the area times the inner [average] circumference [traces around circumference of the circle with radius 9] [so/process] as the top [sweeps hand forward from body] [conclusion].

Some students did not understand the semantic warrant offered by T4 because, for them, the horizontal cross section of the prism could not be represented by a rectangle since the circumference of the inner circle was 16π centimeters and the circumference of the outer circle was 20π centimeters. T3 pressed T4 for explanations and warrants in order to build consensus [00:43:18].

00: 43:18 Tyler: But the circumference on either side of that, like it would still end up being curved because one side would be shorter than the other wouldn't it?
Danielle: Whoa, I didn't get the question.

00:43:33 Paul: The little slices would be so infinitesimally small that it wouldn't matter.
Danielle: Did this make sense when I like un-did this? [unfurls the sheet of paper she had curved into a cylinder]
Heber: No I think what he's saying...
Daniel: Yes
Heber: ...is like if I cut your prism in half, um [Heber walks to the whiteboard and draws two dotted line segments to indicate cutting along the plane of symmetry through the vertex of the parabola]
Danielle: Like right down here?
Paul: Then one side would be longer than the other
00:44:45 Robert: It's like an average.
Mike: Is it the perfect average?
Derrick: Yeah, 'cause we got the same.
Mike: All right, but you didn't prove like that it was...
00:44:52 Heber: Yeah, but you're not showing how the volume of a wrapped prism is going to be the volume of a prism stacked. I just don't, I mean I'm willing to agree that it's true, but I don't see, there's no proof that it's true necessarily.
Tyler: I think what you guys are doing is pretty good cause you're...I could not see it right off cause you didn't say you were averaging it. You just said that you straightened it out.
01:00:00 Heber: Because just saying it works isn't a proof. But I do like their method of solving. It's a cool method.

Substantial Argument and Nested Mathematical Referent: Median of a Trapezoid for Average Radius

Axial coding revealed nested mathematical referents in students' semantic warrant production.

An example of nested mathematical referents is found in a consensus-building response when students from T4 introduced the median of a trapezoid [Figure 7] as backing for their choice to use the circumference of the circle with radius 9 for the height of the prism and to warrant the rectangular horizontal cross section [D3 09:59].

D3 09:59 Derrick: Thanks. 11 minus [writes 11- in the integrand to reflect "top curve minus bottom curve" resulting in $\int_{-1}^{1}\left(11-11x^2\right)dx$]...yeah. my bad. I just forgot. Okay. So we stretched that [the gel-pack] out. That's just like taking the mug and chopping it in half right there [with a chopping motion draws a vertical line segment across the sketched "ring" or "washer" cross-section of the gel-pack] and stretching it out [process/what we have]. And so [draws an average circumference on the "ring" cross-section he had drawn on the whiteboard], so we took this middle radius, the average radius cause this is 8 [draws and labels an inner radius 8] and that's 10 [draws and labels an outer radius 10]. So we took the radius at 9 [so/process]. And we figured out why we did that. Because if you really stretch it out, it looks like this [sketches an isosceles trapezoid with base lengths 8 and 10 units] [because/nested mathematical referent]and there's

space unaccounted for [points to the region on the left of the left leg of the trapezoid]. But if we do it in the middle [draws a segment connecting the midpoints of the legs of the trapezoid] [process], then it makes up for that extra space [draws congruent right triangles using the trapezoid legs as the hypotenuse of each and shades the interior region of the triangles] [then/warrant]. Does that make sense?
Student: Yeah.

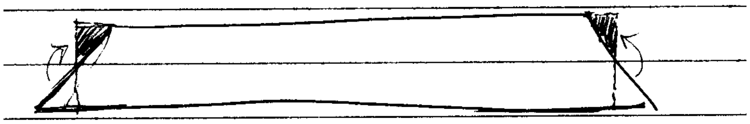

Figure 7. T4's Median of a Trapezoid Nested Mathematical Referent

Students at T4 offered the semantic warrant that the formula for the volume of a prism could be used to find the volume of the gel pack (D2 41:15), in particular, the area of the parabolic face, or base of the prism, multiplied by the circumference of the circle with nine-inch radius, or the height equivalent to an average circumference generated by rotating the parabolic region around the y-axis [Figure 8].

First Theorem of Pappus. If a solid of revolution is generated by revolving a region *R* about a line not cutting the region, then the volume of the solid generated is the product of the area of *R* and the length of the path described by the centroid of *R*. The third dimension characterized by T4 is the length of the path described by the centroid of R in this theorem.

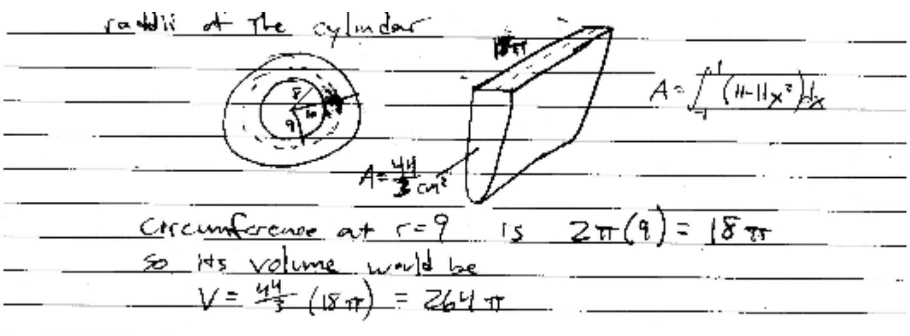

Figure 8. Student work demonstrating the First Theorem of Pappus

Results. All four groups enacted distinct approaches to determine that the gel-pack volume was 264π cubic centimeters. Students chose specific mathematical referents in the production of semantic warrants during public presentations of solution approaches generated during problem solving. T3 chose the prototypical parabola, $y = x^2$, as a referent in their semantic warrant regarding a disc solution comprising integration over an interval of the difference of the volumes of two discs. T4 recognized an isomorphism between the gel-pack and a parabolic prism and chose to use the formula for the volume of a prism as a referent in their semantic warrant regarding an invented solution equivalent to a special case of the First Theorem of Pappus. Patterns emerged in the logical structure of students' substantial arguments. In semantic warrant production, these students flexibly used adverbs, adjectives, conjunctions, pronouns (Figure 9), gestures, and collaboratively created notations to meaningfully relate empirical examples and prior experience-as-given with developmental reasoning and mathematical inferences.

Adverb	**Adjective**	**Conjunction**	**Pronoun**
So (in the way that follows; in the way that precedes)	So (true as stated or reported)	So (in order that; with the result that; on the condition that)	We (nominative plural of I representing a collective viewpoint)
Then (immediately or soon afterward; next in order of time or place; as a consequence)	Like (of the same form; similar, analogous; bearing resemblance	Because or 'cause (for the reason that; due to the fact that)	You (second person singular or plural; people being addressed; synesthetic participation as in "you go up")
		Like (in the same way as, as if)	

Figure 9. Pivotal Cues in Students' Substantial Argumentation

Students' semantic warrant production was grounded in the exercise of personal agency with mindful awareness of others' ways of thinking through temporally extended agential authority. Students were convinced of the reasonableness of multiple solution approaches through semantic warrant production during public performances over time and were strongly influenced by the instructor's introduction of the First Theorem of Pappus after they had invented the theorem in response to mathematical necessity in problem solving. The creativity exhibited by each of the four groups resulted in four distinct approaches to finding the volume of the gel pack. Here, we examined semantic warrant production in public presentations of solution approaches created by two of the groups. Students from Table 4 provided a substantive argument regarding a geometric solution comprising integration to find the area of a parabola forming the base of a prism, revolving the parabolic region around the x-axis and multiplying the area of the parabola by an average circumference generated by the rotation to find volume. Students from Table 3 provided a substantive argument regarding the "disc" solution comprising integration over an interval of the difference of the volumes of two discs. Some students wanted proof that the geometric approach used by Table 4 was valid. An algebraic transformation by the instructor of the solution approach presented by Table 3, resulted in an expression equivalent to the solution approach presented by Table 4, and subsequent introduction by the instructor of the First Theorem of Pappus convinced students that both analogous approaches were valid.

<table>
<tr><th>Substantial Argument</th><th>Syntactic Proof</th></tr>
<tr><td>What We have</td><td>Given</td></tr>
<tr><td>Mathematical Referents</td><td rowspan="4">Definitions/Statements/Theorems</td></tr>
<tr><td>What We Want</td></tr>
<tr><td>So/Process</td></tr>
<tr><td>Semantic Warrants</td></tr>
<tr><td>So/Conclusion</td><td>Deductive Conclusion</td></tr>
</table>

Figure 10. Structural Comparison between Semantic and Syntactic Arguments

We found that mathematical referents played a significant role in the creation of substantial arguments. We also found that substantial arguments have a flexible structure compared to a syntactic proof [Figure 10]. The elements of a substantial argument: what we have, what we want, semantic warrants, mathematical referents, processes and conclusions, do not have to be structured in a designated order to form convincing explanations and justifications.

Implications. Mathematical discourse, creativity, and development of meaning depend on the exercise of personal agency. In this classroom, notational developments and discourse between students depended on essential opportunities to exercise personal agency in offering semantic warrants. Student substantial arguments were clarified in improvisational interplay between student justifications, questions, comments, and co-production of semantic warrants and appeals to mathematical referents to support mathematical inferences. Although justification may be seen as reaction to criticism, justification may also be viewed as a means for building consensus. Students were convinced of the reasonableness of multiple solution approaches through semantic warrant production during public performances over time and were strongly influenced by the introduction of the First Theorem of Pappus *after* they had invented the theorem in response to mathematical necessity in problem solving. Further, pedagogical choices by the instructor to introduce the First Theorem of Pappus in support of students' mathematical work were also based on the exercise of personal agency and were enacted as a means for building on the groundings of student reasoning and mathematical creativity.

We see enacted personal agency as the genesis for creativity and meaning in students' doing of mathematics. Our grounded theory analysis of honors calculus students' substantial arguments during public performances expands our awareness of students' use of pivotal cues, logical structure, mathematical referents and semantic warrants in problem solving and provides insight into students' choices in personal instantiations and justifications that ground mathematical inferences in problem solving. In turn, we begin to recognize more fully how students build convincing justifications for their understandings of mathematical concepts and processes.

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